Journal of Statistical Physics, Vol. 62, Nos. 1/2, 1991

Statistics of Transfer Matrices for Disordered Quantum Thin Metallic Slabs

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Received May 1, 1990

In the quantum transport problem of a tight-binding Anderson model, the statistics of eigenvalues for the transfer matrices of thin disordered slabs is studied. Numerical simulations indicate that the probability distribution of nearest neighbor eigenvalue spacing and the Δ_3 statistics have already become close to that of the Gaussian orthogonal ensemble for sample lengths of the order of the mean free path, provided that transverse localization effects are not important. An intuitive argument is given why this should occur independently of the size of the matrix. Therefore, good mixing of the channels is not essential for obtaining Gaussian orthogonal ensemble type statistics and universal conductance fluctuations.

KEY WORDS: Localization; random matrices; conductance fluctuations.

1. INTRODUCTION

There has been a recent theoretical and experimental interest in conductance fluctuations in normal metallic samples.[1-9] In particular, the statistical variance $\mathrm{var}(g)$ of the conductance g of a set of metallic samples, all with the same macroscopic properties, is independent of the size of the samples and of their average conductance in the metallic regime,

$$\mathrm{var}(g) \sim e^2/\hbar \tag{1}$$

This result has been obtained by perturbation calculations in microscopic theories,[2,3] numerical simulations,[2,9] and macroscopic theories.[5,8] These macroscopic theories are based on the fact that the transfer matrices that determine the conductance, provided a multichannel Landauer formula[10-12] is valid, have spectral properties analogous to the famous

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random matrix ensembles introduced by Wigner and Dyson,[13,14] the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE). The symmetries of the transfer matrices are current conservation and, in the absence of a magnetic field, time reversal. However, in these macroscopic theories, one assumes that the matrices are as random as possible, given the symmetries. This is called a maximum entropy hypothesis.[6,15]

In this paper, we study two-dimensional disordered slabs, whose lengths are not much larger than their widths, and the maximum entropy hypothesis is not valid. We argue that a continuous transition from Poisson-type statistics to GOE-type statistics should occur as the length of the sample increases. This feature should be independent of the width of the sample if transverse localization effects are not important. For lengths larger than a few mean free paths, the statistics has already become GOE type, in agreement with previous numerical simulations of the Anderson model on squares and long slabs.[7,9]

2. METHOD

We consider the usual tight-binding Anderson model for a disordered strip of width N and length n sandwiched between two perfect conducting leads,

$$H = \sum_{i,j,i',j'} \varepsilon_{i,j} |i,j\rangle\langle i,j| + V_{i,j,i',j'} |i,j\rangle\langle i'j'| \tag{2}$$

where (i, j) denotes the site of abscissa i and ordinate j. The summation in Eq. (2) runs over all the sites (i, j) and (i', j'); thus, $1 \leqslant i \leqslant n$, $1 \leqslant i' \leqslant n$, $1 \leqslant j \leqslant N$, and $1 \leqslant j' \leqslant N$. The $\varepsilon_{i,j}$ are random site energies chosen independently in the interval $[-W/2, W/2]$. The hopping elements $V_{i,j,i',j'}$ are equal to 1 if (i, j) and (i', j') are nearest neighbors and zero otherwise. We assume a two-probe measurement with phase breaking processes outside the disordered strip.[16,17] We suppose that we are not in the ballistic regime, $g \ll N$ (n is supposed to be larger than the mean free path l).

A simplified multichannel Landauer formula[12] is assumed to be valid. The conductance g is given by

$$g = 2 \operatorname{Tr} \left(\frac{1}{TT^\dagger + (TT^\dagger)^{-1} + 2} \right) \tag{3}$$

where T is the $2N \times 2N$ transfer matrix of the slab. The transfer matrix is defined by

$$(O', I') = T(I, O) \tag{4}$$

where (I, O) are the incoming and the outgoing fluxes on the left of the sample [(O', I') are the analogous fluxes on the right of the sample]. We write[18,19] T as

$$T = P_1^{-1} \prod_{i=1}^{n} M_i P_0 \tag{5}$$

where P_1 and P_0 are the matrices which project the wave function on the left and on the right of the disordered slab, respectively, onto the propagation modes of the leads. M_i is the usual $2N \times 2N$ transfer matrix for the wave function

$$M_i = \left(\begin{array}{ccccc|c} E-\varepsilon_{i,1} & -1 & 0 & \cdots & \cdots & \\ -1 & E-\varepsilon_{i,2} & -1 & 0 & \cdots & \\ 0 & -1 & E-\varepsilon_{i,3} & -1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & -\mathbb{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \cdots & 0 & -1 & E-\varepsilon_{i,N} & \\ \hline & & \mathbb{1} & & & 0 \end{array}\right) \tag{6}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix. The effect of the coupling to the leads has been studied in detail in ref. 20. We focus on the properties of the transfer matrix of the slab, $\mathscr{T} = \prod_{i=1}^{n} M_i$. From now on, we shall refer to n as the number of iterations. We find it convenient to write $\mathscr{T}$ as $\mathscr{T} = Q^{-1} \mathscr{M} Q$, with

$$Q = \left(\begin{array}{cccccccccccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & & & \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 1 & 0 & & & \\ \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 0 & 0 & & & \\ \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & & & \end{array}\right) \tag{7}$$

$\mathscr{M}$ is a symplectic real matrix with $2n$ nonzero codiagonals.

So far we have been concerned with a particular microscopic model. Macroscopic theories have been developed for properties of filled transfer matrices which are as random as possible (given the symmetries).[6-8,15] However, they apply if two conditions are met, (1) good mixing of the channels, and (2) not being in a localized regime.

In our case, we want to keep the last condition but get rid of the first hypothesis only. We therefore want to have the following conditions:

(a) $n \ll \xi_{1d}(N) \simeq Nl$, where $\xi_{1d}(N)$ is the one-dimensional localization length for a strip of width N.

(b) $N \ll \xi_{1d}(n) \simeq nl$.

(c) $n \ll \xi_{2d}$, where ξ_{2d} is the two-dimensional localization length.

(d) $N \ll \xi_{2d}$.

(plus the condition that we are not in the ballistic regime $l \ll n$).

Since the matrices $\mathcal{M}$ are symplectic real, as a result of current conservation and time reversal, the eigenvalues of $\mathcal{M}\mathcal{M}^\dagger$ are real, positive and go by pairs. If λ is an eigenvalue, $1/\lambda$ is also an eigenvalue. The spectral properties are characterized by the joint probability distribution of eigenvalues lower than 1. We focused on two quantities commonly studied in the statistics of spectra, the probability of nearest neighbor eigenvalue spacing $P(s)$ and the Δ_3 statistics, $\Delta_3(\mathcal{L})$.[13,14]

Before describing our simulations, we give an intuitive argument why $P(s)$ and Δ_3 for our transfer matrices should be practically GOE type, as soon as the number of nonzero codiagonals becomes larger than a certain number, independent of the size of the matrix, provided that localization effects are not important.

The band transfer matrices that we have bear some analogies with an ensemble studied in ref. 21, which considers the matrices whose elements $B_{i,j}$ are given by

$$B_{i,j} = C_{i,j} \exp\left[-\left(\frac{|i-j|}{\sigma} \right)^{\kappa} \right]$$

where $C_{i,j}$ are drawn according to the GOE ensemble, and κ and σ are two real parameters. For this ensemble, they found that the probability of nearest neighbor eigenvalue spacing and the Δ_3 statistics were independent of the size of the matrix (provided that κ is larger than 1), and observed a continuous transition from a close-to-Poisson statistics for small σ to a close-to-GOE statistics for large σ.

In our case, we have a sharp cutoff; the elements of $\mathcal{M}\mathcal{M}^\dagger$ that lie further than $4n$ from the diagonal are zero. The symmetry of our $\mathcal{M}\mathcal{M}^\dagger$

matrices is symplecticity instead of orthogonality. The major difference with ref. 21 is that our matrices have to be symplectic, whereas the matrices of ref. 21 are not strictly orthogonal. Thus, in our case, one should expect more spectral rigidity than in ref. 21.

3. SIMULATIONS AND RESULTS

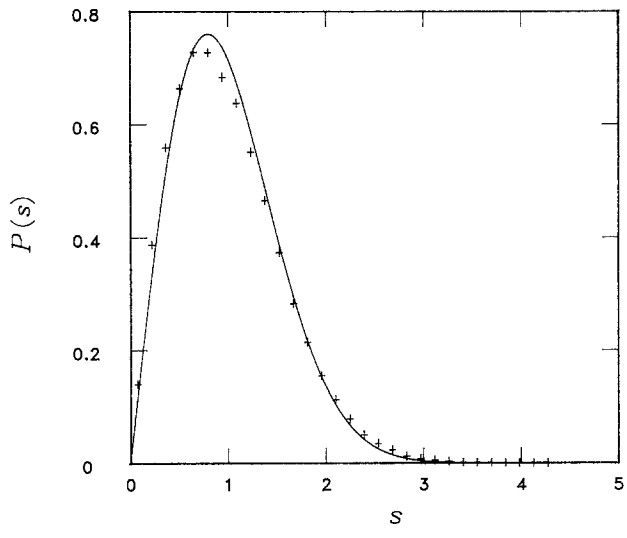
We did our simulations on the usual Anderson model. We studied the probability of nearest neighbor eigenvalue spacing $P(s)$ and the Δ_3 statistics. In order to minimize localization effects, we work in the middle of the band $E=0$ and at weak disorder ($W=0.8$). Roundoff errors can become a serious problem for large matrices. After diagonalization of $TT^{\dagger}$, one checks that the eigenvalues go by pairs $(\lambda, 1/\lambda)$. Since the extreme eigenvalues can differ by several orders of magnitude, it is best to consider[18,19] $\alpha_i = (1/2L)\,\mathrm{Log}\,\lambda_i$, where λ_i is the ith eigenvalue larger than 1. Then, one performs the usual unfolding procedure[22] on the α_i themselves.

Figures 1a–1d show $P(s)$ versus s for $E=0$, $W=0.8$, $N=50$, and $n=5$, 10, 25, 40, respectively. Averages have been taken over 2000 samples, except for $n=40$, where averages are over 200 samples. We also measured $\langle \alpha_i \rangle$, where the brackets mean the average over all samples. The slope $\partial\langle \alpha_i \rangle/\partial(i/N)$ gives an idea[19] of the inverse of the mean free path l. The value of l was found to lie around 10. It is seen that already for $n=5$, the distribution is very close to the GOE result.

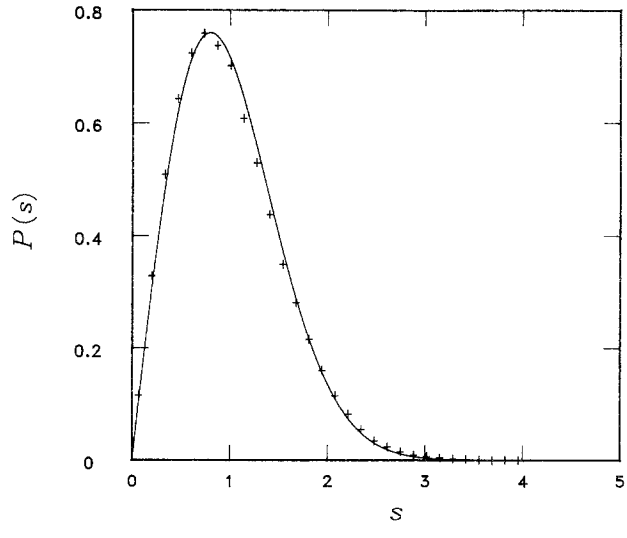
Figures 2a–2d show $\Delta_3(\mathscr{L})$ versus $\mathscr{L}$ for the same samples as in Figs. 1. Apart from the values at $n=5$, the results are not far from the analytical result for the GOE, although there are some deviations.

Thus, for a number of iterations greater than some value n_i, the statistics remains close to GOE, for $N=50$ channels. We want to know how n_i varies as the number of channels increases. From the analogy with the situation of ref. 21 one would expect that n_i does not vary with N.

Figures 3a–3d show $P(s)$ versus s for $n=10$ and $N=10$, 25, 100, and 200, respectively. Except for the $N=200$ curve, the results are close to the GOE. For the Δ_3, shown on Figs. 4a–4d, there are significant deviations from the GOE at $N=100$ and $N=200$. These deviations are presumably due to transverse localization effects, since the localization length for a quasi-one-dimensional strip with ten channels, $\xi_{1d}(10)$, is only 220. Data for $P(s)$ and Δ_3 for $N=50$ are displayed in Figs. 1b and 2b, respectively. The right parameter in the model seems to be the number of nonzero codiagonals of the matrix $\mathscr{M}$, which was $2n$ in our case. Had we taken an Anderson model with next nearest neighbor couplings (with $4n$ nonzero codiagonals in $\mathscr{M}$), then the convergence toward GOE statistics would have been faster as the number of iterations was increased.

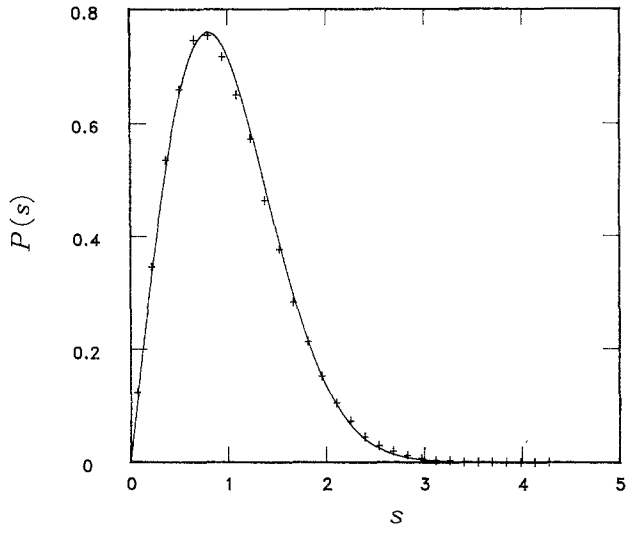


(a)

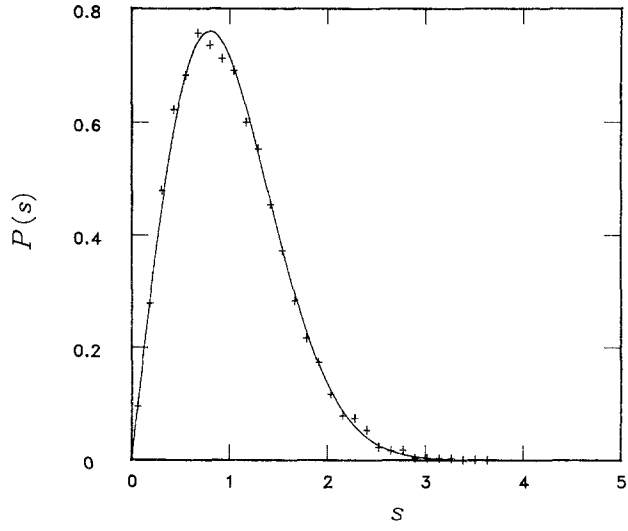


(b)

Fig. 1. Probability of nearest neighbor eigenvalue spacing $P(s)$ versus s for the rescaled spectrum of eigenvalues of the matrix $\mathscr{T}\mathscr{T}^{\dagger}$, where $\mathscr{T}$ is the transfer matrix defined in the text. The values of the parameters are $E=0$, $W=0.8$, width $N=50$, and length (a) $n=5$, (b) $n=10$, (c) $n=25$, (d) $n=40$. Averages have been taken over 2000 samples, except for $n=40$, where averages are over 200 samples. The solid line is the analytic GOE formula.

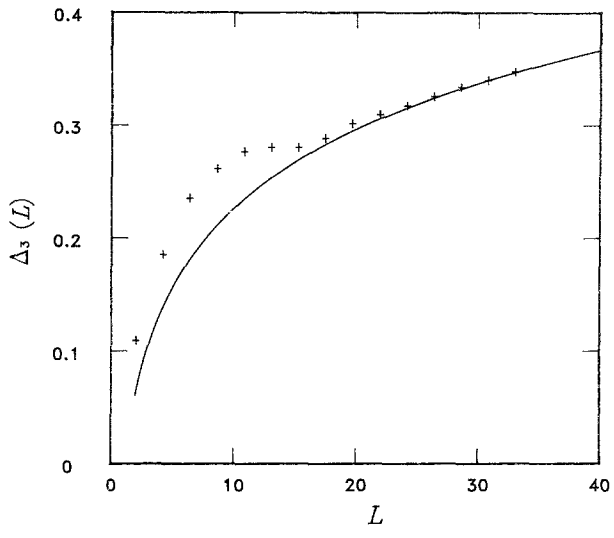


(c)

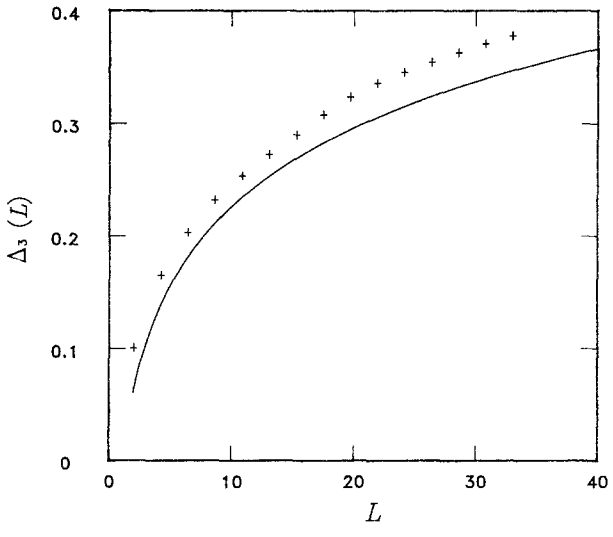


(d)

Fig. 1. (*Continued*)

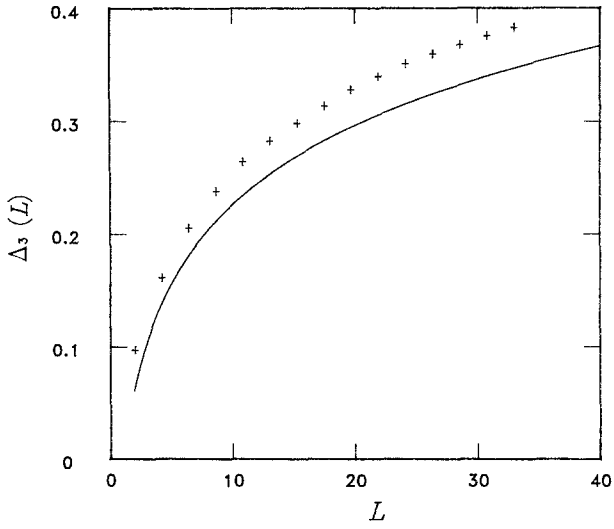


(a)

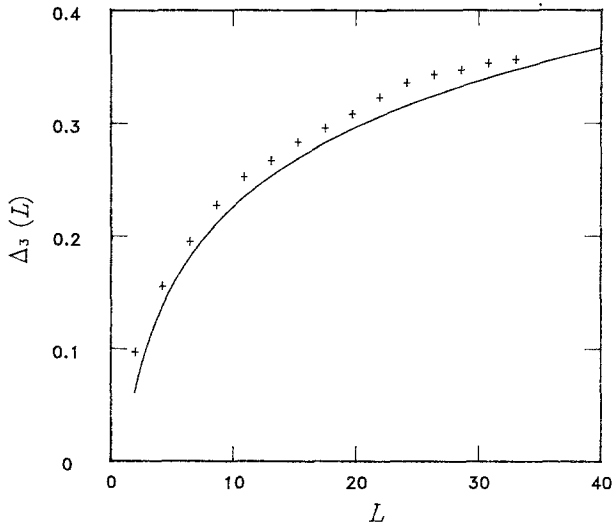


(b)

Fig. 2. Δ_3 statistics $\Delta_3(\mathscr{L})$ versus $\mathscr{L}$, where $\mathscr{L}$ is the average number of eigenvalues in the sequence. For parts (a)–(d), the values of the parameters are the same as in Figs. 1a–1d. The solid line is the analytic GOE formula.

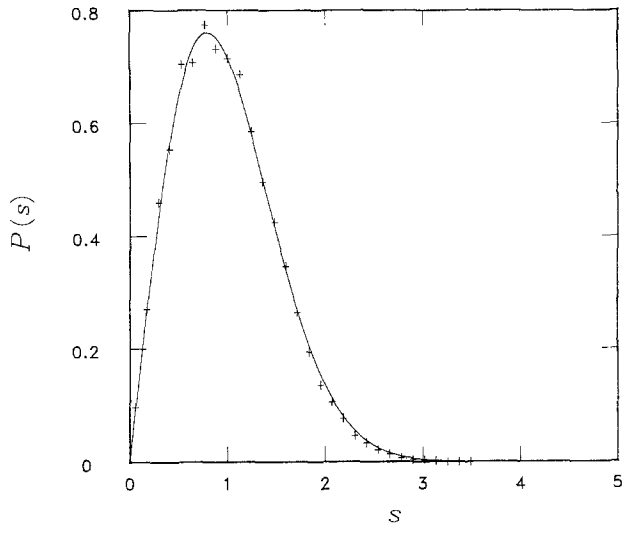


(c)

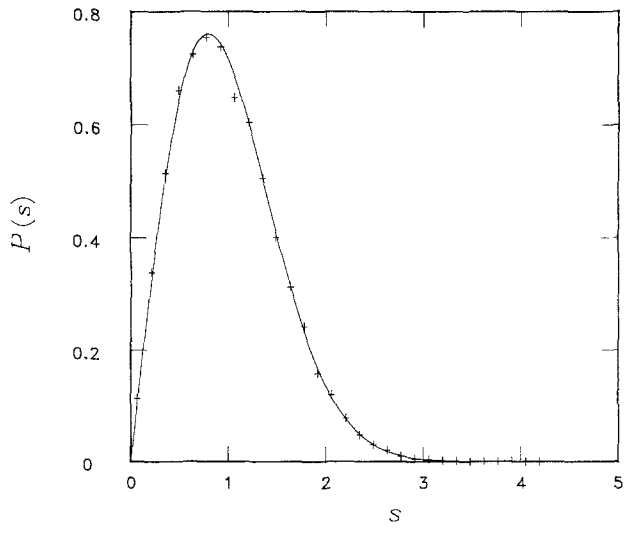


(d)

Fig. 2. (*Continued*)

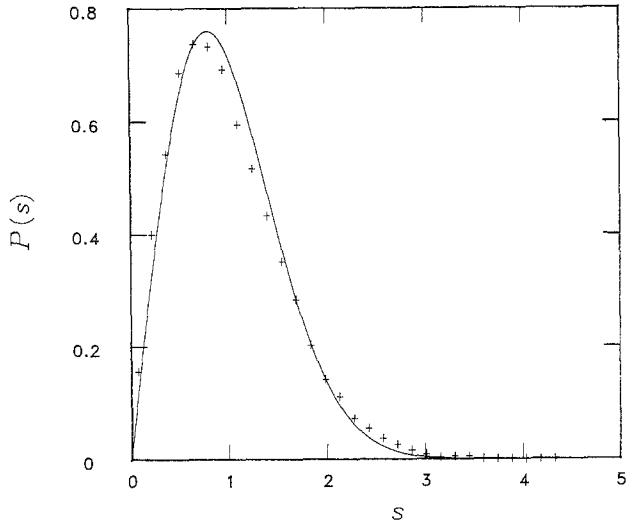


(a)

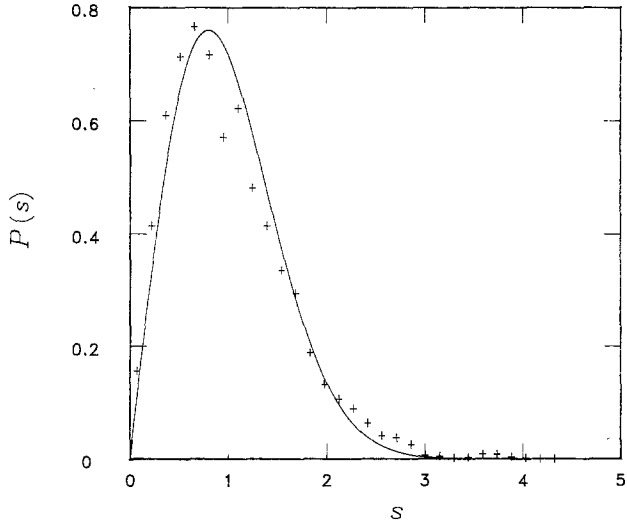


(b)

Fig. 3. Same as in Fig. 1, except that now the length is fixed to $n = 10$ and the width is (a) $N = 10$, (b) $N = 25$, (c) $N = 100$, (d) $N = 200$.

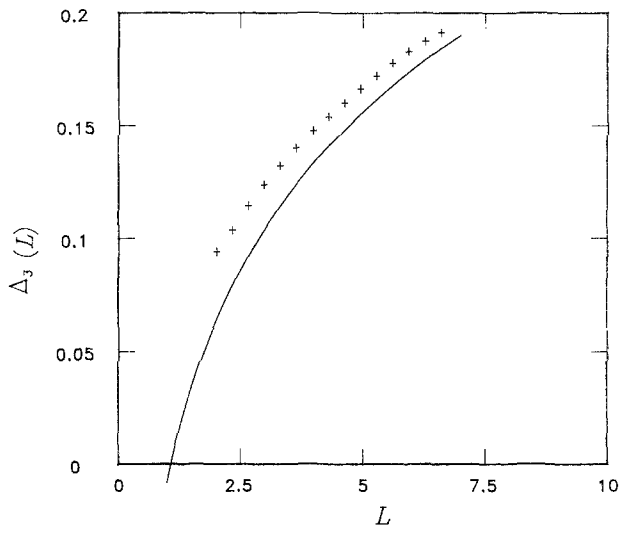


(c)

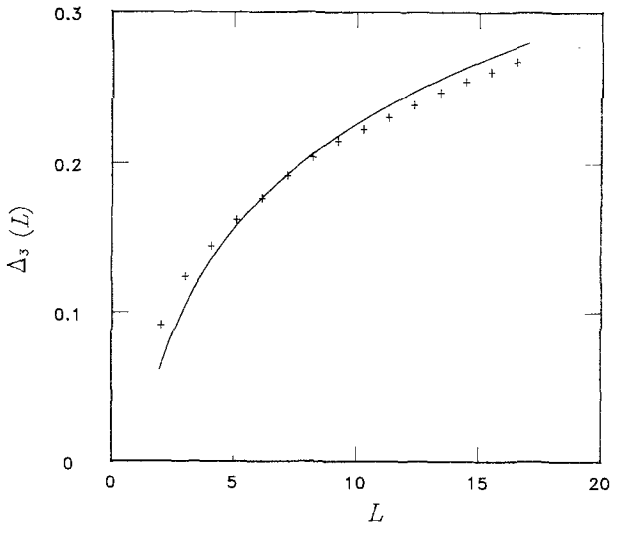


(d)

Fig. 3. (*Continued*)

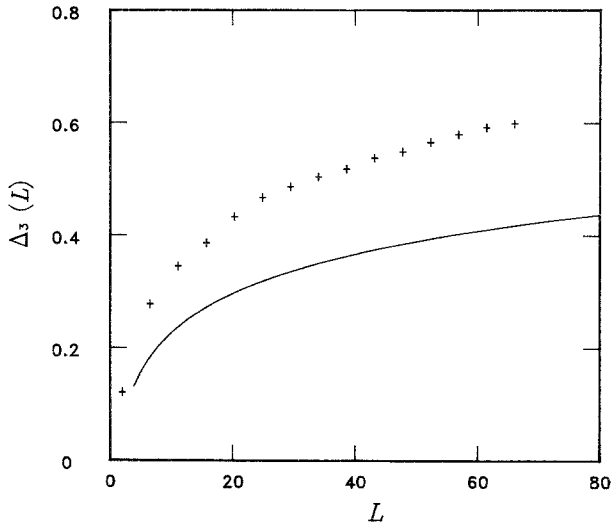


(a)

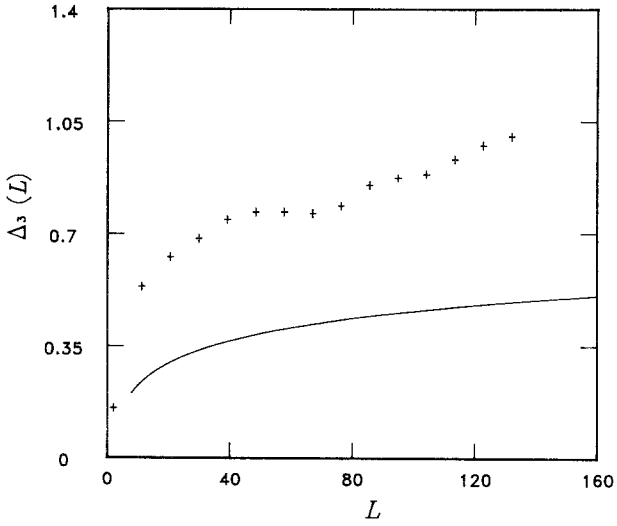


(b)

Fig. 4. Δ_3 statistics, $\Delta_3(\mathscr{L})$ versus $\mathscr{L}$, for the same values of the parameters as in Fig. 3.



(c)



(d)

Fig. 4. (*Continued*)

In summary, we have argued that, for disordered thin slabs, in the metallic regime, good mixing of the channels is not important in order to get GOE-type statistics. Numerical simulations of the Anderson model show that already for ten iterations, i.e., for lengths of the order of the mean free path, the probability distribution of nearest neighbor eigenvalue spacing is very close to the GOE, apparently irrespective of the size of the matrix. For the Δ_3 statistics, there seems to be a slight dependence on the size of the matrix, but this is probably caused by transverse localization effects. We thus expect to have universal conductance fluctuations even if the channels are not well mixed.

ACKNOWLEDGMENTS

We thank J. L. Pichard for stimulating discussion and D. Stauffer for a critical reading of the manuscript.

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